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Planing of a low-aspect-ratio flat ship at infinite Froude number

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SUMMARY

The free-surface elevation caused by the planing motion of a low-aspect-ratio flat ship at infinite Froude number is investigated. A relation is obtained between the physical characteristics of the planing hull and the extent to which it is wetted. Analytic results are presented in the case of laterally-uniform longitudinal hull slope.

1. Introduction

Planing of a boat at high speed on a free surface has been investigated both theoretically and experimentally by a number of authors. A comprehensive bibliography of experimental papers published before 1964 may be found in D. Savitsky's [1] paper on planing hull design.

Early theoretical work was done by Weinblum [2], Wagner [3, 4] and Green [5, 6]. Wagner [3] investigates both two- and three-dimensional planing problems at infinite Froude number and shows that, in his linearised formulation, the flow is equivalent to flow past a thin wing with only its lower side in contact with the fluid. He shows that the splash or spray plume associated with planing problems may be represented in the linearised problem by a square-root singularity in the pressure at the leading edges.

More recent publications include those of Tulin [7], Maruo [8] and Tuck [9]. Tulin [7] considers a slender ship, which he then assumes is also flat, and linearises Laplace's equation and the boundary conditions accordingly. However, to this linearised problem, he adds a spray plume flow at the leading edges. This effect is of second order in the slenderness and, as it is the only effect of this order which is included, it provides an inconsistent solution of the problem, displaying some but not necessarily all second order effects, but correct to first order.

With the exception of Tuck [9], who draws attention to the fact that a "flat plate" does not necessarily imply a rectangular section shape, none of the abovementioned authors are concerned with the shape of either the planing hull or the free surface. Most assume that the complete hull shape, including the shape of the wetted area, is fixed and given. Oertel [10] shows for a flat ship that, if the hull shape is prescribed, the extent to which it is wetted is determined as part of the solution, and conversely.

The present paper is an extension of the low-aspect-ratio flat-ship theory of Tuck [9], and shows how the geometric characteristics of such a hull are inter-related. The formulation of the problem follows that given by Tuck, except that only zero gravity (i.e. infinite Froude number or very high speed) is considered here. General expressions, describing the relationship between the hull characteristics of a planing boat and the free surface elevation outside the boat, are derived for monotone hull shapes. In the special case when the hull slope in the direction of motion is laterally uniform, analytic results are obtained which directly relate the hull slope, hull section shape and waterplane shape.

It is also shown in this particular case that, when $\eta(x, s)$ has positive curvature, the computed pressure vanishes forward of the supposed trailing edge and becomes negative thereafter, suggesting that smooth flow detachment occurs forward of the station of maximum cross-section.

2. Mathematical formulation

A flat ship of low-aspect-ratio is assumed to be moving with speed U in the negative s-direction, the origin of the co-ordinates being fixed to the bow, as shown in Figure 1.

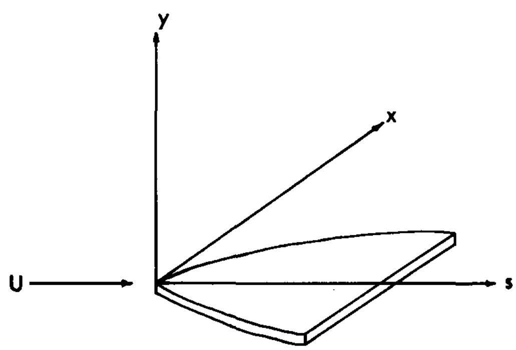

Figure 1. Co-ordinate system.

The surface of the ship is given by

$$y = \eta(x, s) \tag{2.1}$$

for $|x| < b(s)$ and $s < L$, where $b(s)$ is the half-waterplane width at station s and L is the length of the ship. The function η is usually negative, as most of the hull lies below the equilibrium free surface $y = 0$. Outside the hull surface, equation (2.1) defines the free-surface elevation caused by the ship.

Assuming that the flow is irrotational, the velocity field is given by

$$\boldsymbol{q} = \nabla\Phi = \nabla(Us + \phi)$$

where ϕ is the perturbation velocity potential satisfying the full three-dimensional Laplace equation,

$$\phi_{xx} + \phi_{yy} + \phi_{ss} = 0$$

for $y < \eta(x, s)$.

The exact hull boundary condition is

$$\phi_y = (U + \phi_s)\eta_s + \phi_x \eta_x, \tag{2.2}$$

which is applied on the hull surface $y = \eta(x, s)$. Outside the hull surface, equation (2.2) is the kinematic condition on the unknown free surface. For $g = 0$, the dynamic free-surface condition is

$$P/\rho + U\phi_s + \tfrac{1}{2}|\nabla\phi|^2 = 0, \tag{2.3}$$

where P is defined as the excess of pressure over atmospheric at the free surface.

The ship is assumed not only to be flat, but also to be slender, with

$$D \ll B \ll L,$$

where D is the draft and B the beam of the ship. That is, if

$$D = O(\alpha)\cdot L$$

and

$$B = O(\varepsilon)\cdot L,$$

for small parameters α and ε, then

$$\alpha \ll \varepsilon.$$

Making the small-draft approximation, and then the low-aspect-ratio approximation, in equations (2.2) and (2.3) respectively, gives

$$\phi_y = U\eta_s \quad \text{on } y = 0 \tag{2.4}$$

and

$$P/\rho + U\phi_s = 0 \quad \text{on } y = 0. \tag{2.5}$$

The linearised boundary conditions are applied on $y = 0$ because, as $\alpha \to 0$, the hull reduces to its projection on the plane $y = 0$. Since the ship is slender, ϕ is the potential for the cross-flow problem in the (x, y)-plane and satisfies in the limit as $\varepsilon \to 0$ the two-dimensional Laplace equation

$$\phi_{xx} + \phi_{yy} = 0,$$

for $y < 0$.

3. Monotone hull shapes

It will be assumed throughout this and the following section that $\eta(x, s)$ is a strictly monotone-decreasing function of s. The function defining the waterplane, $x = b(s)$, is also

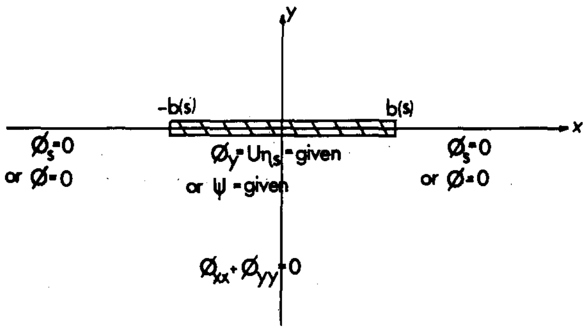

Figure 2. Cross-flow plane.

assumed to be strictly monotonic, so that the flow does not separate from the leading edges of the hull upstream of the trailing edge.

The problem for ϕ in the cross-flow plane is shown in Figure 2. Since the flow is symmetric about $x = 0$, only $x \geq 0$ will be considered.

It is convenient to solve the problem in terms of ϕ and its harmonic conjugate, the stream function, ψ, instead of ϕ_x and ϕ_y. For brevity, $\psi(x, s)$ will be written for $\psi(x, 0, s)$ and similarly for ϕ, ψ_x, etc.

From equation (2.4), the Cauchy–Riemann equations give, assuming ψ is an odd function of x,

$$\psi(x, s) = -U \int_0^x d\xi\, \eta_s(\xi, s) \tag{3.1}$$

on $y = 0$. Thus, ψ is a known function on the projection of the hull onto the plane $y = 0$.

The function $Q(x, s)$ is defined by

$$Q(x, s) = \int_{-\infty}^{s} d\sigma\, P(x, \sigma) = \int_{s_0(x)}^{s} d\sigma\, P(x, \sigma),$$

where $s_0(x)$ is the station at which $x = b(s)$, as $P \equiv 0$ outside the hull projection on $y = 0$. $Q(x, s)$ is the loading on a unit-width strip of the hull at offset x, extending from the leading edge to station s. From equation (2.5) and the definition of Q,

$$Q(x, s) = -\rho U \phi(x, s). \tag{3.2}$$

The Hilbert transform (Tricomi [11], p. 173) of a function f is defined by

$$H_a f(x) = \frac{1}{\pi} \int_{-a}^{a} \frac{d\xi}{x - \xi} f(\xi).$$

When $-a < x < a$, the above integral is interpreted as a Cauchy principal-value integral.

As ψ is known for $|x| < b(s)$, the Plemelj relations (e.g. see Muskhelishvili, [12]) may be used to determine ϕ for $|x| < b(s)$, and hence ψ for $|x| > b(s)$. Thus

$$\psi(x, s) = -H_\infty \phi(x, s) = -H_{b(s)} \phi(x, s),$$

since $\phi \equiv 0$ for $|x| > b(s)$, from equation (3.2) and the definition of Q. Thus,

$$\phi(x, s) = -H_{b(s)}^{-1}\psi(x\ s), \qquad |x| < b(s),$$

$$= (b^2(s) - x^2)^{\frac{1}{2}} H_{b(s)} \frac{\psi(x, s)}{(b^2(s) - x^2)^{\frac{1}{2}}}, \qquad |x| < b(s). \tag{3.3}$$

The operator $H_{b(s)}^{-1}$ is not normally defined uniquely and, to any such solution (3.3), a multiple of $(b^2(s) - x^2)^{-\frac{1}{2}}$ must be added (see Tricomi, [11], p. 174). But only integrable (inverse square-root) velocity and pressure singularities, and hence a square-root zero in ϕ, are allowed at leading edges.

From equation (3.2), the loading on the hull is

$$Q(x, s) = -\rho U(b^2(s) - x^2)^{\frac{1}{2}} H_{b(s)} \frac{\psi(x, s)}{(b^2(s) - x^2)^{\frac{1}{2}}}.$$

To first order in ε, the slenderness parameter, Tulin [7] has solved a problem identical to that solved here. The above result for ϕ agrees to $O(\varepsilon)$ with Tulin's result for the velocity potential, in the special case (to be treated in more detail in the following section) when $\eta_s(x, s)$ is independent of x.

The function ψ may now be determined for $x > b(s)$, namely

$$\psi(x, s) = -H_{b(s)}\phi(x, s)$$

$$= -\frac{1}{\pi}\int_{-b(s)}^{b(s)} \frac{d\xi}{x - \xi}(b^2(s) - \xi^2)^{\frac{1}{2}} \cdot \frac{1}{\pi}\int_{-b(s)}^{b(s)} \frac{dt}{\xi - t}\frac{\psi(t, s)}{(b^2(s) - t^2)^{\frac{1}{2}}}$$

$$= (x^2 - b^2(s))^{\frac{1}{2}} \cdot H_{b(s)} \frac{\psi(x, s)}{(b^2(s) - x^2)^{\frac{1}{2}}}. \tag{3.4}$$

The displacement of the free surface caused by the motion of the ship may now be calculated. Since

$$\eta_s(x, s) = -\psi_x(x, s)/U,$$

equation (3.4) may be used to derive an expression for η_s, namely

$$\eta_s(x, s) = -\frac{1}{(x^2 - b^2(s))^{\frac{1}{2}}} H_{b(s)}\eta_s(x, s)(b^2(s) - x^2)^{\frac{1}{2}}, \qquad x > b(s).$$

So, for $x > b(s)$,

$$\eta(x, s) = -\int_0^s \frac{d\sigma}{(x^2 - b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\eta_\sigma(x, \sigma)(b^2(\sigma) - x^2)^{\frac{1}{2}}. \tag{3.5}$$

To first order in the parameter α/ε, the surface elevation, η, is continuous across $x = b(s)$ (Tulin, [7]). Therefore, for $|x| < b(s)$,

$$\eta(x, s) = -\int_0^{s_0(x)} \frac{d\sigma}{(x^2 - b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\eta_\sigma(x, \sigma)(b^2(\sigma) - x^2)^{\frac{1}{2}} + \int_{s_0(x)}^{s} d\sigma\, \eta_\sigma(x, \sigma) \tag{3.6}$$

Since there is no disturbance in front of the hull, the lower bound of integration is zero in equation (3.5) and in the first integral of equation (3.6).

When $s > L$, $Q(x, s) = Q(x, L)$, so $\phi(x, s)$, $\psi(x, s)$ and hence $\eta_s(x, s)$ are independent of s. Thus,*

$$\eta(x, s) = \eta(x, L) + (s - L)\eta_s(x, L). \tag{3.7}$$

In this case, a discontinuity occurs in the free-surface elevation at $x = \pm b(L)$. As $x \to b(L)_+$ (i.e. from outside the track of the ship), the surface elevation is unbounded (since $\eta_s(x, L)$ is unbounded), whereas, as $x \to b(L)_-$ (i.e. from inside the track of the ship), the surface elevation is finite. Similarly for $x \to -b(L)$. This gives rise to the two lines of white water which may be seen trailing behind a planing boat from its point of maximum beam (which in the present special case necessarily occurs at the trailing edge). Since $g = 0$ in the present model problem, there is no restoring force to damp out the free-surface elevation caused by the planing motion. Thus, as s approaches infinity, $\eta(x, s)$ becomes negatively infinite for $|x| < b(L)$ and positively infinite for $|x| > b(L)$. In practice, gravity is always important at a sufficiently great distance from the planing boat (e.g. see Wu, [13]).

Equation (3.6), rewritten in the form

$$\eta(x, s) = \int_0^s d\sigma\, \eta_\sigma(x, \sigma) + c(x), \tag{3.8}$$

where

$$c(x) = -\int_0^{s_0(x)} \frac{d\sigma}{(x^2 - b^2(\sigma))^{\frac{1}{2}}} \times \frac{1}{\pi}\int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x - \xi}(\eta_\sigma(\xi, \sigma) - \eta_\sigma(x, \sigma))(b^2(\sigma) - \xi^2)^{\frac{1}{2}} - x\int_0^{s_0(x)} \frac{d\sigma}{(x^2 - b^2(\sigma))^{\frac{1}{2}}}\eta_\sigma(x, \sigma), \tag{3.9}$$

shows that the input hull shape (i.e. $\int_0^s d\sigma\, \eta_\sigma(x, \sigma)$) is not the actual wetted hull shape. The true hull shape, given by equation (3.8), is determined by the relationship between the physical properties of the planing hull described in equation (3.9).

If the waterplane shape $b(s)$ and the hull slope $\eta_s(x, s)$ are known, then equation (3.8) uniquely determines the underwater hull shape, i.e. given the waterplane and the hull longitudinal slope, the wetted hull is fully determined. Unfortunately, this is not the problem of greatest practical significance. Usually, the shape of the complete hull (both wetted and

* It is assumed that $\eta(x, s)$ is continuous across the trailing edge.

non-wetted portions) is assumed known and it is the extent of the wetted area, i.e. the function $b(s)$, which is to be determined as part of the solution of the hydrodynamic problem. So, equation (3.9), with $c(x)$ known, is to be considered as an *integral equation* to determine the unknown function $b(s)$. Except in the most simple cases, the direct mathematical problem of finding $b(s)$ is a difficult task, as it involves the solution of an integral equation over a region which is itself unknown. Progress may be possible by adopting a trial-and-error approach using solutions of the indirect problem, i.e. the problem in which $b(s)$ is assumed known. In the following section, a special case in which the direct problem has an explicit solution will be discussed.

4. A planing hull with constant section shape

The results of the last section will now be applied to the case when the hull slope in the s-direction is independent of x, i.e. $\eta_s(x, s) = -f(s)$. This implies that all sections are of the same shape, one section being obtained from another by a vertical translation.

When $s < L$, equations (3.8) and (3.5) respectively give

$$\eta(x, s) = \begin{cases} -\displaystyle\int_0^s d\sigma f(\sigma) + x\int_0^{s_0(x)} d\sigma \frac{f(\sigma)}{(x^2 - b^2(\sigma))^{\frac{1}{2}}}, & x < b(s) \\ \displaystyle\int_0^s d\sigma \frac{f(\sigma)}{(x^2 - b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}(b^2(\sigma) - x^2)^{\frac{1}{2}}, & x > b(s), \end{cases}$$

i.e.

$$\eta(x, s) = \begin{cases} -\displaystyle\int_0^s d\sigma f(\sigma) + x\int_0^{s_0(x)} d\sigma \frac{f(\sigma)}{(x^2 - b^2(\sigma))^{\frac{1}{2}}}, & x < b(s) \\ -\displaystyle\int_0^s d\sigma f(\sigma) + x\int_0^{s} d\sigma \frac{f(\sigma)}{(x^2 - b^2(\sigma))^{\frac{1}{2}}}, & x > b(s). \end{cases}$$

Writing

$$c(x) = x\int_0^{s_0(x)} d\sigma \frac{f(\sigma)}{(x^2 - b^2(\sigma))^{\frac{1}{2}}}, \tag{4.1}$$

$y = c(x)$ is the equation of the hull cross-section shape. Only the vertical position of the section relative to the free surface is controlled by the station co-ordinate s, and is of an amount

$$-\int_0^s d\sigma f(\sigma).$$

When $s > L$, equation (3.7) gives

$$\eta(x, s) = \begin{cases} -\int_0^L d\sigma f(\sigma) - (s-L)f(L) + c(x), & x < b(L) \\ -\int_0^L d\sigma f(\sigma) + x\int_0^L d\sigma \frac{f(\sigma)}{(x^2 - b^2(\sigma))^{\frac{1}{2}}} - (s-L)f(L) & \\ \quad + \frac{x(s-L)f(L)}{(x^2 - b^2(L))^{\frac{1}{2}}}, & x > b(L) \end{cases}$$

where $c(x)$ is defined as in equation (4.1).

If the waterplane shape $b(s)$ and the hull slope $-f(s)$ are known, then equation (4.1) determines the hull cross-section shape function $c(x)$. Conversely, if $b(s)$ is an unknown function, and $c(x)$, $f(s)$ are known then $b(s)$ may be determined uniquely by inverting equation (4.1), as follows.

The substitution

$$\beta = b(\sigma)$$

followed by the transformations

$$\tau = \beta^2$$

and

$$t = x^2$$

yield the equation

$$D(t) = \int_0^t d\tau \frac{G(\tau)}{(t-\tau)^{\frac{1}{2}}}$$

where

$$D(t) = c(t^{\frac{1}{2}})/t^{\frac{1}{2}},$$

$$G(\tau) = F(\tau^{\frac{1}{2}})/2\tau^{\frac{1}{2}}$$

and

$$F(\beta) = f(\sigma)/b'(\sigma).$$

This is in the form of Abel's integral equation (see Tricomi [11], p. 39) and has the unique solution

$$G(t) = \frac{1}{\pi}\frac{d}{dt}\left\{\int_0^t d\tau \frac{D(\tau)}{(t-\tau)^{\frac{1}{2}}}\right\}$$

or

$$\frac{f(s)}{b'(s)} = \frac{2}{\pi}\frac{d}{dx}\left\{\int_0^x d\xi \frac{c(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\right\}, \tag{4.2}$$

where $x = b(s)$. Since $b'(s) = dx/ds$, equation (4.2) gives

$$f(s) = \frac{2}{\pi}\frac{d}{ds}\left\{\int_0^x d\xi \frac{c(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\right\}$$

and so

$$\int_0^s d\sigma f(\sigma) = \frac{2}{\pi}\int_0^x d\xi \frac{c(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}. \tag{4.3}$$

Expressing the left and right sides as $F(s)$ and $G(x)$ respectively, equation (4.3) may be written more simply as

$$F(s) = G(x).$$

Since F is a known function of s and G is a known function of x, x is now a known function of s, identifiable as $x = b(s)$, as required. Thus, the shape of the wetted area may be expressed as a unique function of the hull shape. However, except in the most simple cases, only implicit expressions for x may be obtained.

For example, if

$c(x) = cx^p$, for some real number p $(p > 0)$,

then

$$b(s) = K\left[\int_0^s d\sigma f(\sigma)\right]^{1/p}, \tag{4.4}$$

where

$$K = \pi/cB(\tfrac{1}{2}, (p+1)/2)$$

and $B(u, v)$ is the beta function defined in Gradshteyn and Ryzhik [14]. In particular, in the case of a "flat plate" where $\eta_s(x, s)$ is constant, then the waterplane has the same shape as the hull cross-section. For example, a triangular cross-section implies necessarily a triangular waterplane. Oertel [10] shows that, for a flat ship of finite span, a triangular waterplane produces an approximately V-shaped hull, which becomes more exact as the aspect ratio decreases. This result has also been used by Savitsky [1]. From equation (4.4), if $p = 1$, i.e. the cross-section is triangular, then the waterplane has the same shape as the input hull, i.e. $\int_0^s d\sigma f(\sigma)$.

Figure 3 shows some results for other cross-section shapes and hull slopes. It is easily seen from this table that, as the hull cross-section becomes more cusped, so does the shape of the wetted area and that the rate at which it becomes more cusped relative to that of the hull cross-section depends directly on the power of s in $\eta_s(x, s)$.

When $b(s)$ and $c(x)$ are known, equation (4.2) is an expression determining $\eta_s(x, s)$ uniquely. It is clear, therefore, that there is a direct relationship between hull cross-section shape, waterplane shape and hull slope in the s-direction and that, given any two, the third is predetermined.

For the particular case of a low-aspect-ratio wedge, i.e.

$$\eta_s(x, s) = -\alpha, \qquad |x| < b(s),$$

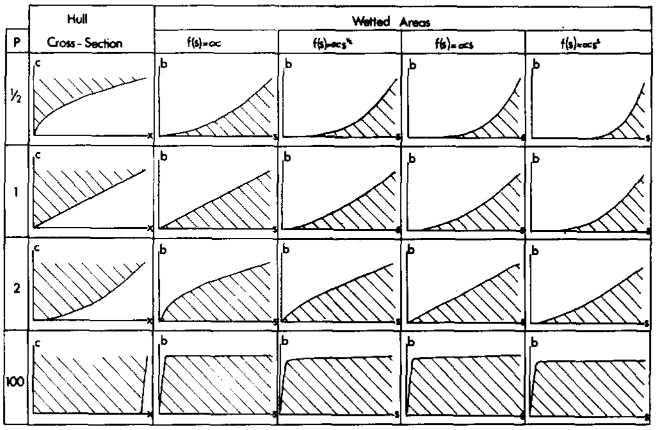

Figure 3. Wetted area shapes for given hull section shapes and hull slopes.

and

$$c(x) = cx,$$

where α and c are small constants, the above results simplify considerably. From equation (4.4),

$$b(s) = \pi\alpha s/2c = \beta s, \text{ say.}$$

When $s < L$,

$$\eta(x, s) = \begin{cases} -\alpha s + \alpha x\, \pi/2\beta, & x < \beta s \\ -\alpha s + \alpha x \arcsin(\beta s/x)/\beta, & x > \beta s. \end{cases}$$

The shape of the free surface for $s =$ constant is shown in Figure 4. At $x = \beta s$, η_x is infinite, but $\eta(\beta s, s)$ is finite and $\eta = O(x^{-2})$ as x approaches infinity. $\eta(x, s) = 0$ when $x = 2\beta s/\pi$. Thus, the actual wetted width of the planing hull is $\pi/2$ times the wetted width measured in the absence of the uniform stream, in agreement with Wagner's [3] results.

When $s > L$,

$$\eta(x, s) = \begin{cases} -\alpha s + \alpha x\, \pi/2\beta, & x < \beta L \\ -\alpha s + \alpha x \arcsin(\beta L/x)/\beta + \dfrac{(s - L)\alpha x}{(x^2 - \beta L^2)^{\frac{1}{2}}}, & x > \beta L. \end{cases}$$

The shape of the free surface for $s =$ constant is shown in Figure 5. Note that the free-surface elevation is now theoretically infinite along $x = b(L)$, as discussed earlier.

A more realistic shape for a planing hull is one with a chine, i.e. a discontinuity in $\eta_x(x, s)$. Such a shape is shown in Figure 6. In order that the results obtained in this section concerning waterplane shape may be applied to this problem, the point $x = B$ at which the discontinuity occurs must remain constant with respect to s, i.e. the chine must occur along a fixed offset x.

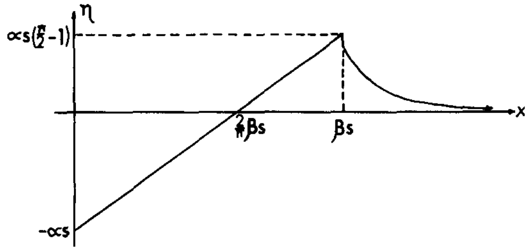

Figure 4. Free-surface shape for $s < L$.

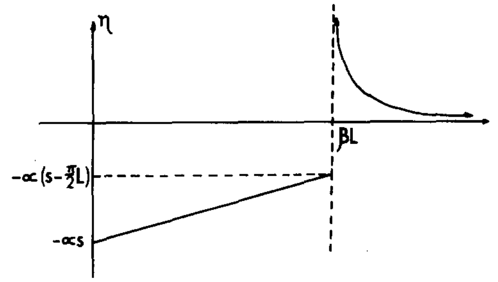

Figure 5. Free-surface shape for $s > L$.

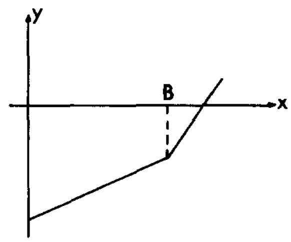

Figure 6. Cross-section of a chine.

When $s < L$ and $x < b(s)$, it is supposed that

$$\eta(x, s) = \begin{cases} -\displaystyle\int_0^s d\sigma f(\sigma) + c_1(x), & 0 < x < B \\ -\displaystyle\int_0^s d\sigma f(\sigma) + c_2(x), & B < x < b(s), \end{cases}$$

where c_1 and c_2 are known functions of x and f is a known function of σ, but $b(s)$ is yet to be determined. $c_1(B)$ and $c_2(B)$ need not be equal, as shown in Figure 6, as the results are still valid for the case of a positive jump of $O(\alpha)$ in $c(x) = c_1(x) + c_2(x)$ at $x = B$. However, any $c(x)$ which produces a non-monotone $b(s)$ is not permissible, since separation will occur from a non-monotone waterplane shape and the problem is considerably altered. Indications are that if $c(x_1) = c(x_2)$, where $x_1 < B$ and $x_2 > B$, i.e. $c_1(x_1) = c_2(x_2)$, then $b(s)$ will be non-monotone. When $s < s_0$, where $b(s_0) = B$,

$$\eta(x, s) = -\int_0^s d\sigma f(\sigma) + c_1(x), \qquad x < b(s).$$

Once $b(s)$ is known, η may be determined outside the hull by using equations (3.5) and (3.7).

From equation (4.3).

$$\int_0^s d\sigma f(\sigma) = \frac{2}{\pi} \int_0^x d\xi \frac{c_1(\xi)}{(x^2 - \xi^2)^{\frac{1}{2}}}, \qquad \text{when } x < B, \tag{4.5}$$

and

$$\int_0^s d\sigma f(\sigma) = \frac{2}{\pi}\int_0^B d\xi \frac{c_1(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}} + \frac{2}{\pi}\int_B^x d\xi \frac{c_2(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}, \quad \text{when } x > B. \tag{4.6}$$

From these two equations, the function defining the waterplane shape, $x = b(s)$, may be determined. It is clear that these results may be readily generalised to the case of a finite number of discontinuities in $\eta_x(x, s)$.

When an implicit relation for x is obtained, the waterplane shape may be determined as follows. If equations (4.5) and (4.6) are rewritten as

$$F(s) = G(x), \tag{4.7}$$

a graph of $G(x)$ vs. x is drawn. From equation (4.7), this is also a graph of $F(s)$ vs. x. Since $F(s)$ is a known function of s, the vertical axis may be rescaled to give a graph of s vs. x, i.e. the waterplane shape. For example, if $\eta_s(x, s) = -\alpha$, then $F(s) = \alpha s$ and the graph immediately gives the waterplane shape multiplied by a constant. Further examples are given in Figure 7.

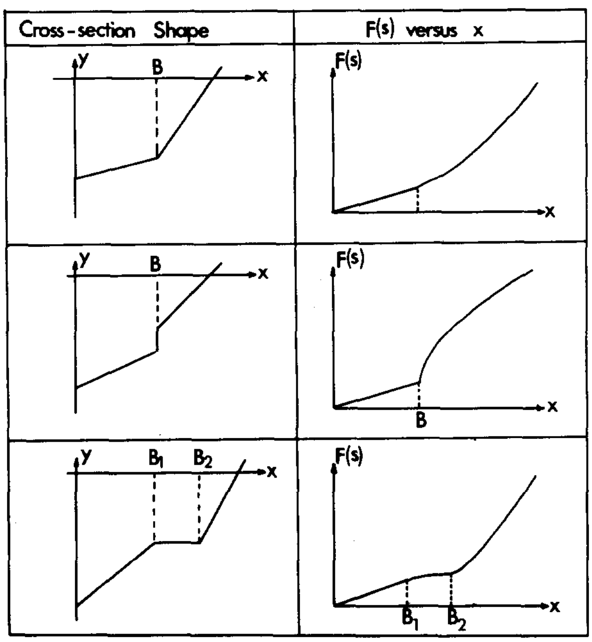

Figure 7. Determining the waterplane shape for a given section shape.

5. The occurrence of negative pressures

In obtaining the results of Section 3, it was assumed that the trailing edge was at station $s = L$. Hence, the wetted length, L, along the keel line was fixed and, given any underwater hull shape, the waterplane shape, described by the function $x = b(s)$, was the only undetermined characteristic of the hull. In this section, it will be shown that, in some cases, this assumption cannot be made.

When $\eta_s(x, s) = -f(s)$, the pressure, $P(x, s)$, is given by

$$P(x, s) = \rho U^2 \left(f'(s)(b^2(s) - x^2)^{\frac{1}{2}} + \frac{f(s)b(s)b'(s)}{(b^2(s) - x^2)^{\frac{1}{2}}} \right). \tag{5.1}$$

Therefore, along the keel line, $x = 0$,

$$P(0, s) = \rho U^2 (f'(s)b(s) + f(s)b'(s)). \tag{5.2}$$

For all the hull shapes of this kind considered so far, the right-hand side of equation (5.2) is always positive. But, if $f'(s) < 0$, it is possible for this expression to vanish at some station, say $s = s_m$, forward of $s = L$ and to be negative for $s > s_m$. Since negative pressure is unacceptable, it must be assumed that separation has actually taken place forward of L, and that the portion of the hull between $s = s_m$ and $s = L$ is not wetted (c.f. Oertel, [10]).

For example, suppose the hull has a parabolic keel profile and triangular sections, i.e.

$$f(s) = 2\alpha(L - s), \qquad s < L$$

and

$$c(x) = cx,$$

where α and c are small positive constants. Note that $s = L$ is the point of maximum draft, and the assumption of a monotone-increasing hull form is satisfied. Then,

$$f'(s) = -2\alpha$$

and is negative everywhere. From equation (4.2),

$$b(s) = \pi\alpha s(2L - s)/2c$$

and, from equation (5.2),

$$P(0, s) = \pi\rho U^2 \alpha^2 (4L^2 - 12Ls + 6s^2)/2c. \tag{5.3}$$

At $s = L$, $f(s)b'(s) = 0$, but $f'(s)b(s) < 0$. Hence, if equation (5.2) did describe the actual pressure for all stations s along $x = 0$, then $P(0, L)$ would be negative. Thus, the flow must have separated forward of $s = L$. In fact, from equation (5.3), $P(0, s) = 0$ when $s = s_m(0) = L - L/\sqrt{3}$, which is 57.7% forward of L.

From equation (5.1), $P(x, s)$ vanishes on the curve $s = s_m(x)$, where

$$x = \pi\alpha(3s_m^4 - 12s_m^3 L + 14s_m^2 L^2 - 4s_m L^3)^{\frac{1}{2}}/2c$$

and is negative for points (x, s) downstream of this curve. This suggests that the flow separates from the underside of the hull along the given curve and that the waterplane has the shape shown in Figure 8. However, it can only be conjectured that the curve $s = s_m(x)$ of Figure 8 defines the true trailing edge, since the results of Section 3 cannot be assumed to be correct past $s = s_m(0)$. To determine what really happens for $s > s_m(0)$, a new Riemann–

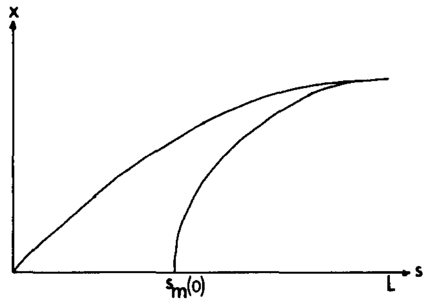

Figure 8. A possible waterplane shape for parabolic keel lines.

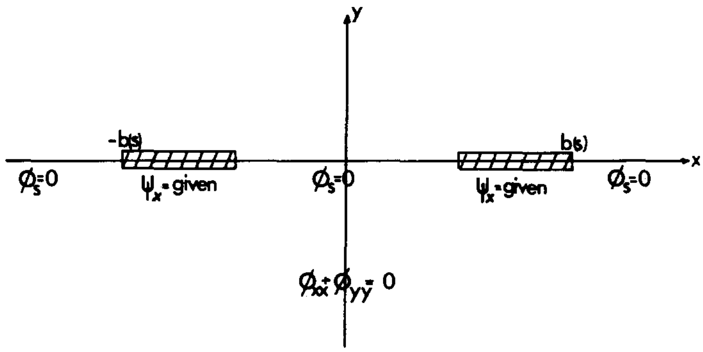

Figure 9. The modified Riemann–Hilbert problem for $s > s_m(0)$.

Hilbert problem, shown in Figure 9, must be solved. This corresponds for $s > s_m(0)$ to a waterplane of the general form of Figure 8, but with a separation curve $s = s_m(x)$ whose shape is to be determined. Work is proceeding on this problem.

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